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Classical, quantum and total correlations

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Abstract
We discuss the problem of separating consistently the total correlations in a bipartite quantum state into a quantum and a purely classical part. A measure of classical correlations is proposed and its properties are explored.

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In quantum information theory it is common to distinguish between purely classical information, measured in bits, and quantum information, which is measured in qubits. These differ in the channel resources required to communicate them. Qubits may not be sent by a classical channel alone, but must be sent either via a quantum channel which preserves coherence or by teleportation through an entangled channel with two classical bits of communication [1]. In this context, one qubit is equivalent to one unit of shared entanglement, or 'e-bit', together with two classical bits. Any bipartite quantum state may be used as a communication channel with some degree of success, and so it is of interest to determine how to separate the correlations it contains into a classical and an entangled part. A number of measures of entanglement and of total correlations have been proposed in recent years [2–6]. However, it is still not clear how to quantify the purely classical part of the total bipartite correlations. In this paper we propose a possible measure of classical correlations and investigate its properties.

We first review the existing measures of entangled and total correlations. In classical information theory, the Shannon entropy, $H(X) \equiv H(p) = -\sum_i p_i \log p_i$, is used to quantify the information in a source, X, that produces messages x_i with probabilities p_i [7, 8]. The relative entropy is a useful measure of the closeness of two probability distributions $\{p_i\}$ and $\{q_i\}$ from the same source X. The relative entropy of $\{p_i\}$ to $\{q_i\}$ is defined as $H(p\|q) = \sum_i p_i \log \frac{p_i}{q_i}$. Correlations between two different random variables X and Y are measured by the mutual information, $H(X : Y) = H(X) + H(Y) - H(X, Y)$, where $H(X, Y) = -\sum_{i,j} p_{ij} \log p_{ij}$ is the joint entropy and p_{ij} is the probability of outcomes x_i and y_j both occurring. The mutual information measures how much information X and Y have in common. It may also be defined as a special case of the relative entropy, since it is a measure of how distinguishable a joint probability distribution p_{ij} is from the completely uncorrelated pair of distributions $p_i p_j$, $H(p_{ij}\|p_i p_j) = H(p_i) + H(p_j) - H(p_{ij})$.

In a quantum context, the results of a measurement $\{E_y\}$ on a state represented by a density matrix, ρ, comprise a probability distribution $p_y = \mathrm{Tr}(E_y\rho)$. The Von Neumann entropy is a way of measuring the information in a quantum state by taking the entropy of the probability distribution generated from the state ρ by a projective measurement onto the state's eigenvectors [9]. It is defined as $S(\rho) = -\mathrm{Tr}(\rho\log\rho) = H(\lambda)$, where $\lambda = \{\lambda_i\}$ are the eigenvalues of the state. The classical relative entropy and classical mutual information also have analogues in the quantum domain. The quantum relative entropy of a state ρ with respect to another state σ is defined as $S(\rho\|\sigma) = -S(\rho) - \mathrm{Tr}(\rho\log\sigma)$. The joint entropy $S(\rho_{AB})$ for a composite system ρ_{AB} with two subsystems A and B is given by $S(\rho_{AB}) = -\mathrm{Tr}(\rho_{AB}\log\rho_{AB})$ and the Von Neumann mutual information between the two subsystems is defined as

$$I(\rho_{A:B}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

As in the classical case, the mutual information is the relative entropy between ρ_{AB} and $\rho_A\otimes\rho_B$. The mutual information is usually used to measure the total correlations between the two subsystems of a bipartite quantum system.

The entanglement of a bipartite quantum state ρ_{AB} may be quantified by how distinguishable it is from the 'nearest' separable state, as measured by the relative entropy. Relative entropy of entanglement, defined as

$$E_{\mathrm{RE}}(\rho_{AB}) = \min_{\sigma_{AB}\in D} S(\rho_{AB}\|\sigma_{AB})$$

has been shown to be a useful measure of entanglement (D is the set of all separable or disentangled states) [4, 5]. Note that $E_{\mathrm{RE}}(\rho_{AB}) \leqslant I(\rho_{A:B})$, by definition of $E_{\mathrm{RE}}(\rho_{AB})$, since the mutual information is also the relative entropy between ρ_{AB} and a completely disentangled state, $I = S(\rho_{AB}\|\rho_A\otimes\rho_B)$ and so must be higher than the minimum over all disentangled states.

Another way to measure the entanglement of a bipartite quantum state is to consider the process of formation of an ensemble of entangled states [2]. The ensemble is first prepared locally by Alice, then one subsystem is compressed using Schumacher compression [10] and sent to Bob by teleportation. The entanglement of formation is then the amount of entanglement required for the teleportation. For pure states this is given by the compression efficiency, $E_{\mathrm{F}}(\rho_{AB}) = S(\rho_B)$. For mixed states, the entanglement of formation is $E_{\mathrm{F}}(\rho_{AB}) = \min\sum_i p_i S(\rho_B^i) \leqslant S(\rho_B)$, where the minimum is taken over all decompositions of the mixed state. However, teleportation which requires only this much entanglement must be accompanied by classical communication of information about the decomposition of the mixed state [11]. The information in the subsystem $S(\rho_B)$ is thus split into a classical part and a quantum part. The classical part may be transmitted by a classical channel, but the quantum part requires entanglement and is sent by teleportation.

There has been some work on the general problem of splitting information in a particular quantum state into a classical and a quantum part [12]. Consider performing a general measurement on the state, $A_iA_i^\dagger$, such that $\rho_B^i = \frac{A_i\rho_BA_i^\dagger}{\mathrm{tr}(A_i\rho_BA_i^\dagger)}$. The final state of subsystem B is then $\sum_i A_i\rho_BA_i^\dagger = \sum_i p_i\rho_B^i$. The entropy of the residual states is $\sum_i p_iS(\rho_B^i)$. The classical information obtained by measuring outcomes i with probabilities p_i is $H(p)$. If the states ρ_B^i have support on orthogonal subspaces, then the entropy of the final state is the sum of the residual entropy and the classical information $S(\sum_i p_i\rho_B^i) = H(p) + \sum_i p_iS(\rho_B^i)$. It has been shown that the state $\rho_B = \sum_i p_i\rho_B^i$ can be reconstructed with arbitrarily high fidelity from the classical measurement outcomes and the residual states if and only if the residual states ρ_B^i are on orthogonal subspaces [12]. We see then that the information in a quantum state may be split into a quantum and a classical part.

We now ask how this can be done for correlations between two subsystems. We would like a way to measure the classical correlations between two subsystems. We first suggest four reasonable properties we should expect a measure of classical correlations, C, to satisfy:

(i) $C = 0$ for $\rho = \rho_A \otimes \rho_B$. This requires that product states are not correlated.
(ii) C is invariant under local unitary transformations. This is because any change of basis should not affect the correlation between two subsystems.
(iii) C is non-increasing under local operations. If the two subsystems evolve *independently* then the correlation between them cannot increase.
(iv) $C = S(\rho_A) = S(\rho_B)$ for pure states. This is a natural requirement, as we will see below.

Note that (ii) and (iv) are also required of a measure of entanglement. If classical communication were added to (iii), it would be identical to the corresponding condition for entanglement measures. However, if classical communication is allowed, then the classical correlations could increase as well as decrease, which is not satisfactory. It is also natural that the measure C should be symmetric under interchange of the subsystems A and B. This is because it should quantify the correlation between subsystems rather than a property of either subsystem. However, we do not include this as a separate constraint as it is not clear that this condition is independent of (i)–(iv).

We now suggest a measure which satisfies these properties. The proposed measure is

$$C_B(\rho_{AB}) = \max_{B_i^\dagger B_i} S(\rho_A) - \sum_i p_i S\left(\rho_A^i\right) \tag{1}$$

where $B_i^\dagger B_i$ is a POVM performed on the subsystem B and $\rho_A^i = \mathrm{tr}_B(B_i \rho_{AB} B_i^\dagger)/\mathrm{tr}_{AB}(B_i \rho_{AB} B_i^\dagger)$ is the remaining state of A after obtaining the outcome i on B. Alternatively,

$$C_A(\rho_{AB}) = \max_{A_i^\dagger A_i} S(\rho_B) - \sum_i p_i S\left(\rho_B^i\right) \tag{2}$$

if the measurement is performed on subsystem A instead of on B. Clearly $C_A(\rho_{AB}) = C_B(\rho_{AB})$ for all states ρ_{AB} such that $S(\rho_A) = S(\rho_B)$. It remains an open question whether this is true in general. The measure is a natural generalization of the classical mutual information, which is the difference in uncertainty about the subsystem B (A) before and after a measurement on the correlated subsystem A (B), $H(A : B) = H(B) - H(B|A)$. Similarly, equations (1) and (2) represent the difference in Von Neumann entropy before and after the measurement. Note the similarity of the definition to the Holevo bound which measures the capacity of quantum states for classical communication [13].

The following example provides an illustration. Consider a bipartite separable state of the form

$$\rho_{AB} = \sum_i p_i |i\rangle\langle i|_A \otimes \rho_B^i$$

where $\{|i\rangle\}$ are orthonormal states of subsystem A. Clearly the entanglement of this state is zero. The best measurement that Alice can make to gain information about Bob's subsystem is a projective measurement onto the states $\{|i\rangle\}$ of subsystem A. Therefore the classical correlations are given by

$$C_A(\rho_{AB}) = S(\rho_B) - \sum_i p_i S\left(\rho_B^i\right).$$

For this state, the mutual information is also given by

$$I(\rho_{A:B}) = S(\rho_B) - \sum_i p_i S\left(\rho_B^i\right).$$

This is to be expected since there are no entangled correlations and so the total correlations between A and B should be equal to the classical correlations.

We now investigate the properties of the quantities in equations (1) and (2). Property (i) above is clearly satisfied, since the state of subsystem B corresponding to any measurement result i on subsystem A is still ρ_B for a product state. In fact, $C(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$. Property (ii) is satisfied since the Von Neumann entropy is invariant under local unitary transformations. Property (iv) is also satisfied since for pure states $C_A(\rho_{AB}) = S(\rho_A)$ $(C_B(\rho_{AB}) = S(\rho_B) = S(\rho_A) = C_A(\rho_{AB}))$ can always be achieved by local projection onto the Schmidt basis. Therefore for pure states $E(\rho_{AB}) = C(\rho_{AB})$ and $I(\rho_{A:B}) = 2E(\rho_{AB}) = 2C(\rho_{AB})$ (here $E(\rho_{AB})$ may be either $E_{\mathrm{RE}}(\rho_{AB})$ or $E_{\mathrm{F}}(\rho_{AB})$ since these measures coincide for pure states). The most important property required of a measure of classical correlations is that it is non-increasing under local operations (property (iii)). We now show that this property is satisfied by the proposed measure.

Property (iii). The measure $C_A(C_B)$ is non-increasing under local operations.

Proof. Let $\{A_i^\dagger A_i : \sum_i A_i^\dagger A_i = I\}$ be the POVM which maximizes

$$C_A = \max_{A_i^\dagger A_i} S(\rho_B) - \sum_i p_i S\big(\rho_B^i\big) = \max_{A_i^\dagger A_i} \sum_i p_i S\big(\rho_B^i \| \rho_B\big).$$

(a) Consider a local operation ϕ_A on subsystem A. This may be regarded as part of the POVM on A so C_A, being a maximum, is not affected.
(b) Now take a local operation ϕ_B on subsystem B. Then by the property that the relative entropy does not increase under local operations [14]

$$\sum_i p_i S\big(\rho_B^i \| \rho_B\big) \geqslant \sum_i p_i S\big(\phi_B\big(\rho_B^i\big) \| \phi_B(\rho_B)\big).$$

Therefore C_A does not increase under local operations. □

We now consider the relations between the classical, total and entangled correlations in some simple cases. These raise some interesting general questions.

First, consider a maximally entangled pure state, $|\phi^+\rangle\langle\phi^+|$, and the family of states that interpolate between it and its completely decohered state $|00\rangle\langle 00| + |11\rangle\langle 11|$. These are states of the form

$$\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + (1-p)|\phi^-\rangle\langle\phi^-|$$

where $\frac{1}{2} \leqslant p \leqslant 1$. The mutual information as a function of p is $I(\rho_{A:B}) = 2 + p\log p + (1-p)\log(1-p)$. The entanglement is $E_{\mathrm{RE}}(\rho_{AB}) = 1 + p\log p + (1-p)\log(1-p)$ [5]. However the classical correlations remain constant at $C_A(\rho_{AB}) = C_B(\rho_{AB}) = C(\rho_{AB}) = 1$. This is achieved by a projective measurement onto $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, and must be the maximum because C cannot exceed one. For this example, the total correlations are just the sum of the entangled and the classical correlations, $I(\rho_{A:B}) = E_{\mathrm{RE}}(\rho_{AB}) + C(\rho_{AB})$, see figure 1.

We now consider a Werner state of the form

$$\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + \frac{1-p}{4} I$$

with $\frac{1}{2} \leqslant p \leqslant 1$. The mutual information is given by

$$I(\rho_{A:B}) = 2 + f\log f + (1-f)\log\left(\frac{1-f}{3}\right)$$

where $f = \frac{3p+1}{4}$. The relative entropy of entanglement is $E_{\mathrm{RE}}(\rho_{AB}) = 1 + f\log f + (1-f) \times \log(1-f)$ [5]. The state is symmetric under interchange of subsystems A and B, so

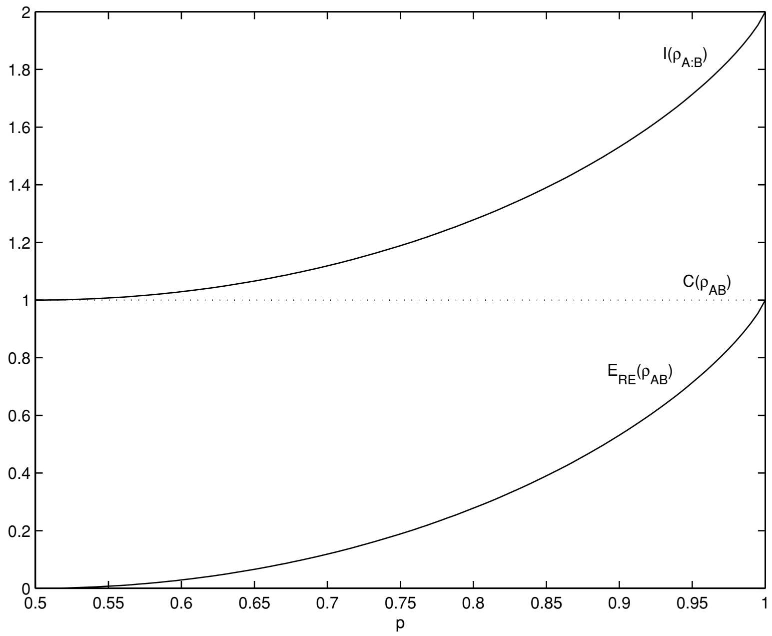

Figure 1. Correlations for a mixture of two Bell states, $\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + (1-p)|\phi^-\rangle\langle\phi^-|$, as a function of p.

$C_A(\rho_{AB}) = C_B(\rho_{AB}) \equiv C(\rho_{AB})$. Any orthogonal projection produces the same value for the classical correlations. We call this quantity $C_p(\rho_{AB})$. Clearly $C_p(\rho_{AB}) \leqslant C(\rho_{AB})$. These quantities are plotted in figure 2.

Consider now a state of the form

$$\rho_{AB} = p|0\rangle|0\rangle\langle 0|\langle 0| + (1-p)|+\rangle|+\rangle\langle +|\langle +|.$$

Again, the state is symmetrical with regard to A and B, so $C_A(\rho_{AB}) = C_B(\rho_{AB}) \equiv C(\rho_{AB})$. This state provides a simple example where the states on both sides are non-orthogonal. In this case, the optimal single-shot measurement for distinguishing the two states $|0\rangle$ and $|+\rangle$ with respect to probability of error is known [15]. However, interestingly it is not the measurement which optimises the classical correlations. We optimize over all orthogonal measurements and call the resulting quantity $C_p(\rho_{AB})$. This is plotted in figure 3, together with the mutual information.

In these last two examples, we see that $C_p(\rho_{AB}) + E_{\mathrm{RE}}(\rho_{AB}) < I(\rho_{A:B})$. If the classical correlations are maximized by an orthogonal measurement on one subsystem, $(C_p(\rho_{AB}) = C(\rho_{AB}))$, the classical and entangled correlations do not account for all the total correlations. This may indicate that the quantum mutual information is not the best quantity for measuring total correlations, or that correlations are simply not additive in this sense. However, $C_p(\rho_{AB})$ may not coincide with $C(\rho_{AB})$. It is also possible that an asymptotic measurement on many copies of the state would achieve a higher value for the classical correlations than measurements on a single copy. This is because the classical correlations are super-additive, $C(\rho\otimes\rho) \geqslant 2C(\rho)$. It is interesting to note that, on the other hand, entangled correlations, as measured by E_{RE} or E_{F}, are subadditive, $E(\rho\otimes\rho) \leqslant 2E(\rho)$, and total correlations, measured by the mutual information, are additive $I(\rho\otimes\rho) = 2I(\rho)$.

A number of interesting questions are raised about the general relations between I, E and C. We do not know whether the sum of the two types of correlations is generally greater

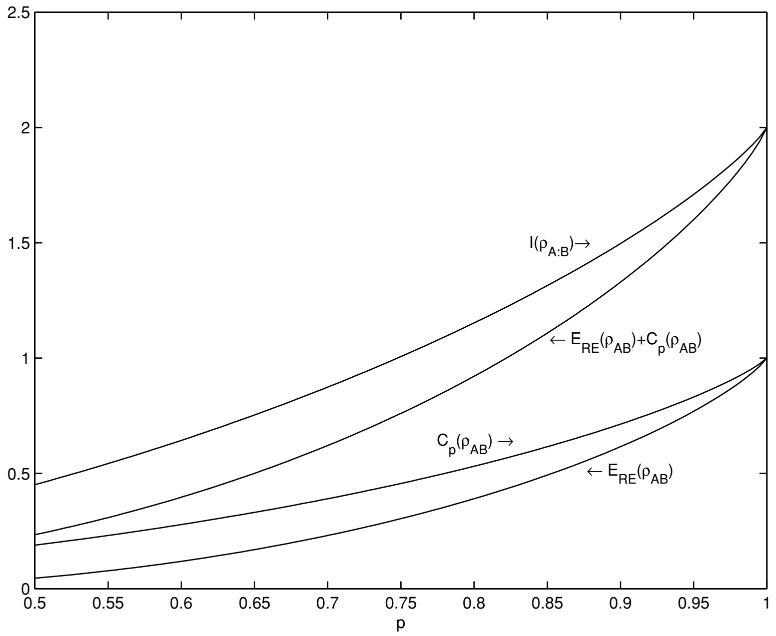

Figure 2. Correlations for a Werner state, $\rho_{AB} = p|\phi^+\rangle\langle\phi^+| + \frac{1-p}{4}I$, as a function of p.

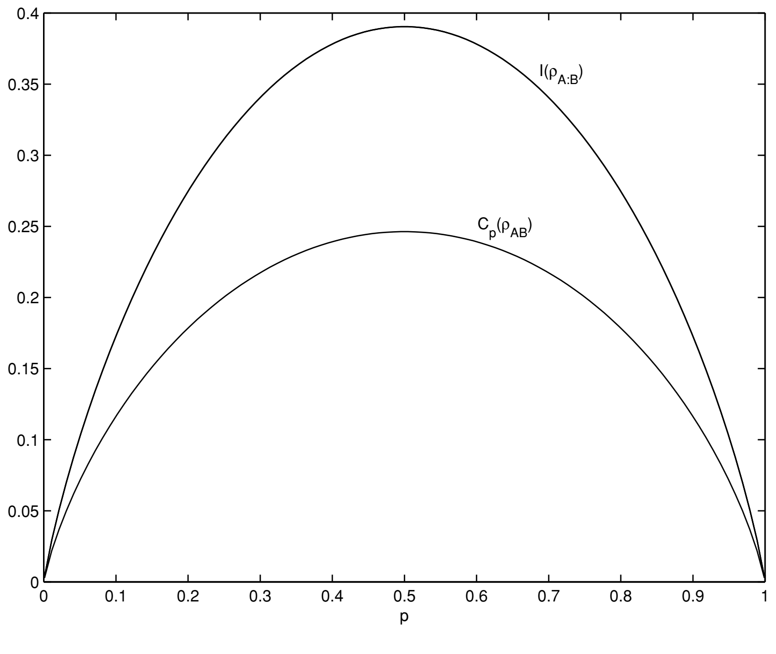

Figure 3. Correlations for the separable state, $\rho_{AB} = p|0\rangle|0\rangle\langle 0|\langle 0| + (1-p)|+\rangle|+\rangle\langle +|\langle +|$, as a function of p.

than, less than or equal to the total correlations, when asymptotic measurements are taken into account. For mixed states, we saw that it need no longer be true that $E(\rho_{AB}) = C(\rho_{AB})$, as it is for pure states. This raises the question of whether $E(\rho_{AB}) = C(\rho_{AB})$ if and only if ρ_{AB} is pure. In our examples we found $E(\rho_{AB}) \leqslant C(\rho_{AB})$, and we conjecture that this is generally true. We know that $E_{\rm RE}(\rho_{AB}) \leqslant I(\rho_{A:B})$. Is it also true that $C(\rho_{AB}) \leqslant I(\rho_{A:B})$ in general?

Another possible measure of classical correlations could be based on the relative entropy, just as measures of total and entangled correlations are both relative entropies, $I(\rho_{A:B}) = S(\rho_{AB}\|\rho_A \otimes \rho_B)$, and $E(\rho_{A:B}) = \min_{\sigma_{AB}\in D} S(\rho_{AB}\|\sigma_{AB})$ [4, 5]. Classical correlations could then be given by the relative entropy between the closest separable state, σ^*_{AB}, and the product state $\rho_A \otimes \rho_B, C_{\rm RE} = S(\sigma^*_{AB}\|\rho_A \otimes \rho_B)$. For the example of a mixture of two Bell states, $C_{\rm RE}(\rho_{AB})$ coincides with $C(\rho_{AB}) = 1$. For the separable state $\rho_{AB} = p|0\rangle|0\rangle\langle 0|\langle 0| + (1-p)|+\rangle|+\rangle\langle +|\langle +|, C_{\rm RE}(\rho_{AB}) = I(\rho_{A:B})$, which makes sense intuitively since there is no entanglement. However, for Werner states, the relative entropy of classical correlations remains constant at $C_{\rm RE}(\rho_{AB}) = 0.2075$. Therefore for low values of p, $C_{\rm RE}(\rho_{AB}) > E_{\rm RE}(\rho_{AB})$, whereas for high values, $C_{\rm RE}(\rho_{AB}) < E_{\rm RE}(\rho_{AB})$. In general $I(\rho_{A:B}) > C_{\rm RE}(\rho_{AB}) + E_{\rm RE}(\rho_{AB})$, so that the two types of correlations do not sum to the total. It also remains to be proved whether $C_{\rm RE}$ is non-increasing under local operations.

In this paper we have raised the question of how to quantify the purely classical part of a correlated quantum system, and we have suggested a potential candidate for a measure which satisfies the most important property of being non-increasing under local operations. A number of interesting open questions about the relationship between measures of classical, entangled and total correlations have been raised. It is hoped that a quantitative understanding of the different types of correlations might aid our understanding of protocols involving manipulation of entanglement and classical information. In particular it should shed some light on the conversion from entanglement to classical information which occurs in the process of quantum measurement.

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